On k-Nearest Neighbor Voronoi Diagrams in the Plane

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***Abstract*—The notion of Voronoi diagram for a set of N points in the Euclidean plane is generalized to the Voronoi diagram of order k and an iterative algorithm to construct the generalized diagram in $0(k^2N \log N)$ time using $0(k^2(N - k))$ space is presented. It is shown that the k-nearest neighbor problem and other seemingly unrelated problems can be solved efficiently with the diagram.**

***Index Terms*—Analysis of algorithm, computational complexity, divide and conquer technique, k-nearest neighbors, point location, Voronoi diagram.**

I. INTRODUCTION

THE nearest neighbor problem arises in several applications such as density estimation, pattern classification, and information retrieval. The problem is to find, among a set of points (or feature vectors), the one which is most similar or closest to a given *test point* according to some dissimilarity or distance measure. One straightforward way of solving it is to compute the "distances" between each point of the set and the test point and then search for the point P with minimum distance. The amount of work involved in this method is obviously proportional to N, the size of the set. If only a few queries will be posed on a given set of data, the method is probably the best one. But on the other hand, if an extremely large number of queries is to be made, then it would be cost effective to perform some moderately elaborate preprocessing on the data. In the latter case, a solution to the given problem is usually evaluated by the following three measures: 1) the *search time*, that is, the number of operations required to find the desired point or points, 2) the *preprocessing time*, that is, the number of operations required to construct the data structure postulated by the search algorithm, and 3) the amount of *storage* required by the preprocessed data structure.

The k-nearest neighbor problem to be considered here is a variant of the classical nearest neighbor problem. It has been shown that the k-nearest neighbor approach is an important technique for multivariate density estimation [18] and classification [5]. In an information retrieval system, the request of searching in a file with N records for the k nearest records to a query becomes more desirable. Due to the time-consuming process of finding the k-nearest neighbors, a number of data structures and search algorithms have been developed [1], [2], [4], [7]-[10] to facilitate a rapid solution. However, the algorithms which appeared in the literature have worst case performance in terms of running time $0(N)$. Therefore, we are interested in the possibility of discovering a technique whose worst case performance is provably faster. From here on, we shall consider only the k-nearest neighbor problem in the Euclidean plane.

To solve the k-nearest neighbor problem, Shamos and Hoey [22] propose an approach using Voronoi diagrams. They introduce the idea of generalized Voronoi diagram in the Euclidean plane, called Voronoi diagram of order k, and claim without proof that the number of regions in a Voronoi diagram of order k never exceeds $0(k(N - k))$. In the following sections we shall prove the bound and give an algorithm to construct the diagram in $0(k^2N \log N)$ time using $0(k^2(N - k))$ storage. In the next section we shall define the notion of Voronoi diagrams. In Section III we prove some properties of the generalized Voronoi diagram. In Section IV, we present a basic procedure to construct the generalized Voronoi diagram. In Section V the search algorithms are summarized. Finally, applications and possible further studies of this topic are discussed.

II. THE VORONOI DIAGRAM AND ITS GENERALIZATION

Given a set $S = \{q_1, q_2, \cdots, q_N\}$ of N sample points in the plane R^2 in which each point q_i is represented as an ordered pair (x_i, y_i), $i = 1, 2, \cdots, N$, the distance $d_P(q_i, q_j)$ between two points q_i and q_j under the L_p-metric is defined as $d_p(q_i, q_j) = (|x_i - x_j|^p + |y_i - y_j|^p)^{1/p}$ and $d_\infty(q_i, q_j) = \max(|x_i - x_j|, |y_i - y_j|)$. We shall consider only the Euclidean metric, i.e., L_2-metric. The locus of points closer to q_i than to q_j, denoted by $h(q_i, q_j)$, is one of the half-planes determined by the bisector $B(q_i, q_j) = \{r | d(q_i, r)^1 \leq d(q_j, r)\}$. The locus of points closer to q_i than to any other point in S, denoted by $\mathcal{V}(i)$, is thus given by $\mathcal{V}(i) = \bigcap_{i \neq j} h(q_i, q_j)$, the intersection of all the half-planes associated with q_i. Vertices of the Voronoi polygon are called *Voronoi points* and the edges on the boundary are called *Voronoi edges.* The set of Voronoi polygons partitions the plane into N regions, some of which may be unbounded, and is referred to as the Voronoi diagram $V(S)$ for the set S

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[1] The subscript $p = 2$ for the metric will be omitted.

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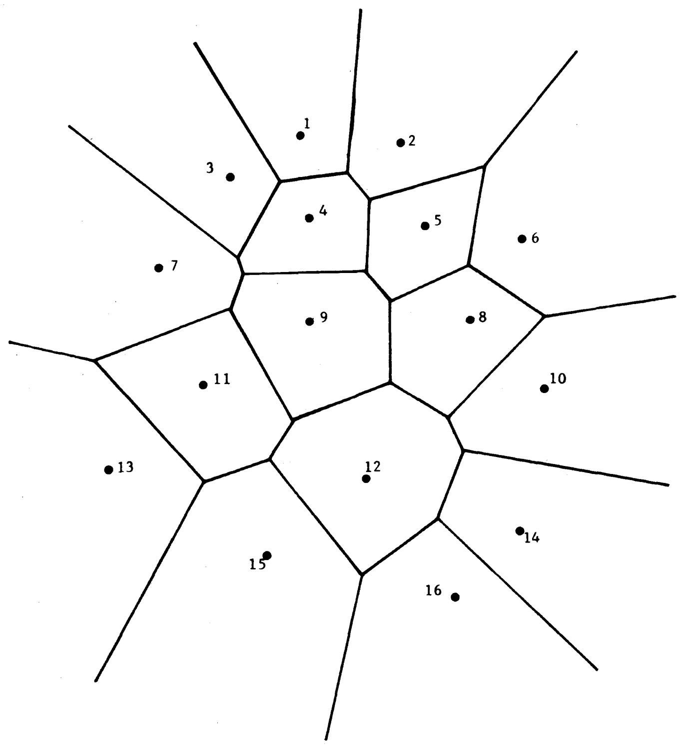

Fig. 1. The nearest neighbor Voronoi diagram for a set of 16 points.

of N points [20], [21]. Fig. 1 is the Voronoi diagram for a set of 16 points. It has been shown that the Voronoi diagram for a set of N points in the plane under the L_p-metric can be constructed in $0(N \log N)$ time [14].

In an ordinary Voronoi diagram each polygon is associated with a point, i.e., $\mathcal{V}(i)$ is associated with q_i. We now extend the notion by considering the Voronoi polygon $\mathcal{V}(H)$ associated with a subset H of points. That is, $\mathcal{V}(H)$ is the locus of points closer to all points in H than to any other point not in H. Or equivalently, $\mathcal{V}(H) = \bigcap_{q_i \in H, q_j \in S-H} h(q_i, q_j)$, where S denotes the entire set of points. For simplicity we use $\cap h(H, S-H)$ to denote $\bigcap_{q_i \in H, q_j \in S-H} h(q_i, q_j)$. Since H can be any of the 2^N possible subsets of S, $\mathcal{V}(H)$ may, of course, not exist. We shall restrict ourselves to the case that the subset H under consideration has cardinality k. We define the Voronoi diagram of order k, denoted $V_k(S)$, as the collection of Voronoi polygons of all the k-subsets of S, namely, $V_k(S) = \cup \{\mathcal{V}(H): H \subset S, |H| = k\}$ [13], [22]. Since some of the polygons $\mathcal{V}(H)$ may be empty, the total number of polygons in $V_k(S)$ depends on the positions of the given N points. As we shall show below, the number is $0(k(N-k))$. Before we proceed, we shall make the following observations which are the basis for the iterative construction of $V_k(S)$ from $V_1(S)$. We assume that no more than 3 points of the given set S are cocircular.

Suppose that $V_1(S)$ has been determined [14], [22]. After locating the test point in the diagram, we can determine its nearest neighbor. The second nearest neighbor can only be one of the points whose associated Voronoi polygons are adjacent to the one in which the test point lies. Thus, if we can somehow "partition" the Voronoi polygon where the test point lies into a number of subregions each of which is adjacent to one of its surrounding Voronoi polygons, the location of the test point in the subregions will identify its second nearest neighbor. If we carry out the "partitioning" on each Voronoi polygon in $V_1(S)$, we shall end up with $V_2(S)$. Similarly, by partitioning $V_2(S)$ we shall obtain $V_3(S)$, and so on. Thus, with $(k-1)$ iterations of this procedure we can obtain from $V_1(S)$ the order k Voronoi diagram $V_k(S)$. The k-nearest neighbors of a given point can be determined by simply locating the test point in $V_k(S)$. Note that the final $V_k(S)$ is useful only for fixed k, assumed to be known *a priori*. We shall discuss the case where k can vary over a certain range later.

Let us now illustrate the idea by an example. Fig. 2 shows an order 1 Voronoi diagram on eight points. Suppose that the test point q lies in $\mathcal{V}(5)$. Its nearest neighbor is q_5. To find its second nearest neighbor, we may artificially "delete" q_5 and solve the nearest neighbor problem as if there were seven points. The Voronoi diagram on the remaining seven points will then divide the polygon $\mathcal{V}(5)$ into six subregions as shown by thick lines in Fig. 2 and the point associated with the subregion in which q lies is the second nearest neighbor. Fig. 3 shows the final configuration of $V_2(S)$ when the transformation is performed on all eight polygons. Fig. 4 shows $V_3(S)$ on the same set of points.

If we superimpose $V_2(S)$ and then $V_3(S)$ on the original $V_1(S)$ diagram, shown, respectively, in Figs. 5 and 6, we see that there are essentially two types of polygons in the generalized Voronoi diagram (to be shown later). Regions of the first kind, referred to as type I regions, contain a single edge of the Voronoi diagram of one lower order. Regions of the second kind, referred to as type II regions, contain a Voronoi point which existed in the previous two lower order Voronoi diagrams. To be specific, given an order 1 Voronoi diagram $G_1 = (I_1, E_1)$, where I_1 is the set of Voronoi points and E_1 the

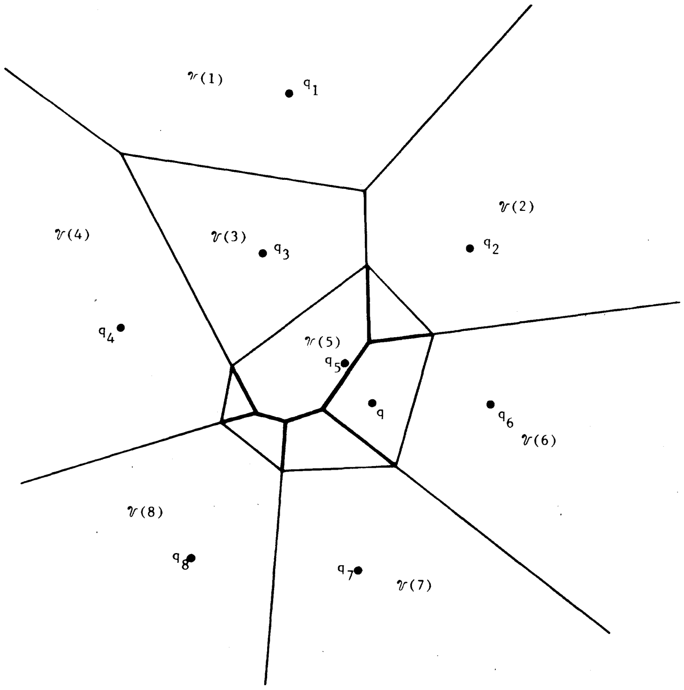

Fig. 2. The order 1 Voronoi diagram for a set of 8 points with a test point in 𝒱(5).

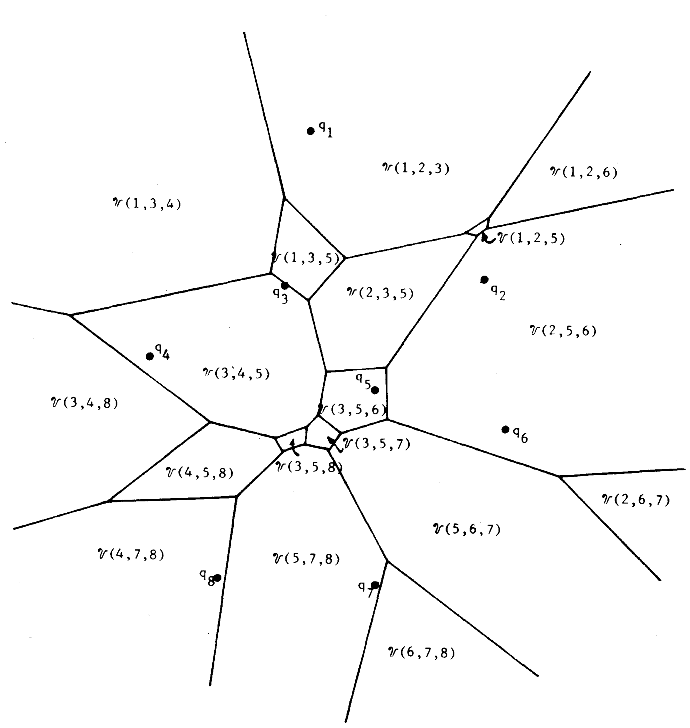

Fig. 4. The order 3 Voronoi diagram for the set of 8 points in Fig. 2.

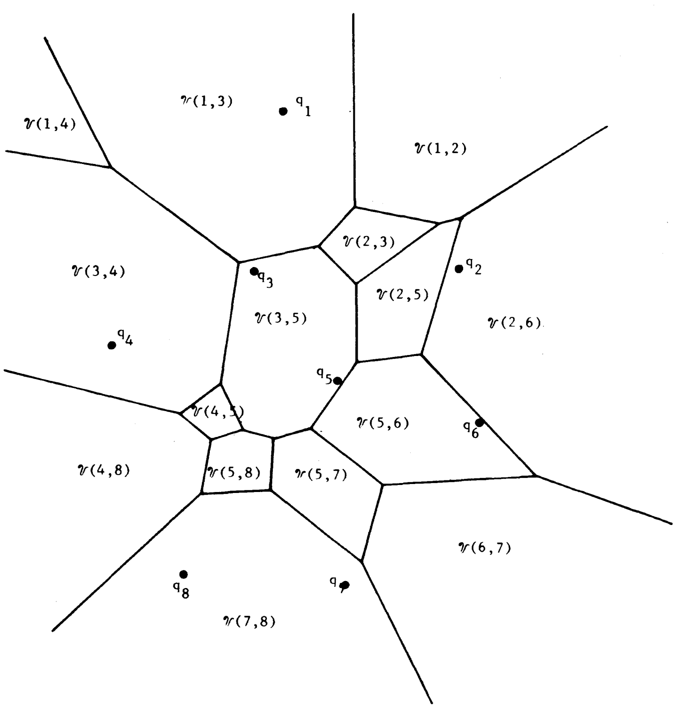

Fig. 3. The order 2 Voronoi diagram for the set of 8 points in Fig. 2.

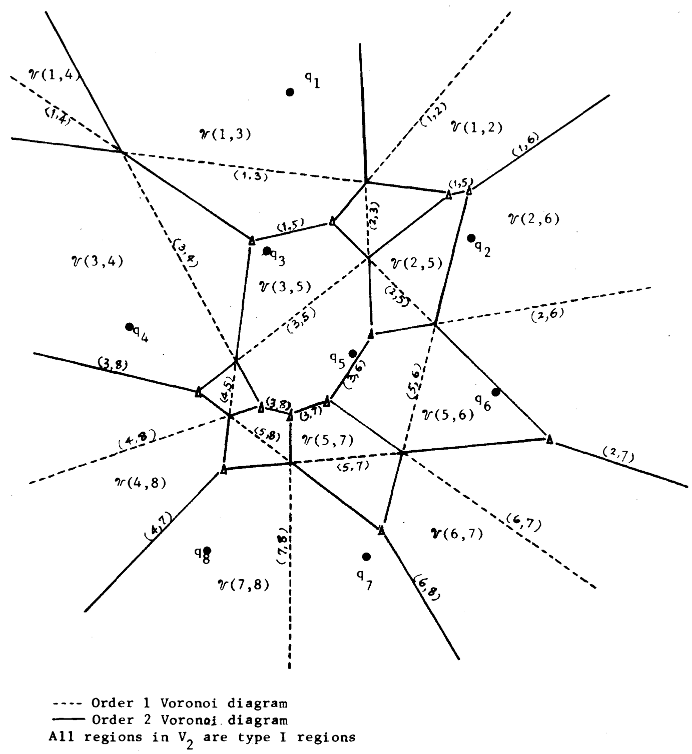

Fig. 5. The order 2 and order 1 Voronoi diagrams superimposed.

set of edges, the order 2 Voronoi diagram turns out to be $G_2 = (I_2, E_2)$, where $I_2 = I_1 \cup I_2'$ is the union of the "old" set I_1 and a "new" set I_2' and E_2 is obtained from E_1 of G_1 by the transformation described earlier. The edge in E_2 which connects two new points in I_2' is called a *new edge*. Thus, the set E_2 consists of new edges and those incident upon "old" points in I_1. In Fig. 5 new points are denoted by "Δ." The pair of indices (i, j) associated with each edge means that the edge is a portion of the bisector of q_i and q_j. Where space permits, every region is represented by $\mathcal{V}(i, j)$ indicating the 2 associated points q_i and q_j. To be consistent, the region should be denoted by $\mathcal{V}(\{i, j\})$. With no confusion, we shall omit the set symbol { } in the following discussion. As in a set, the order of the indices is irrelevant. There are only regions of one type, namely type I regions in V_2 shown in Fig. 5, since every edge in V_1 is new. On the other hand, there are two types of regions in V_3 (Fig. 6), each of which is associated with 3 indices. For instance, $\mathcal{V}(1, 3, 5)$ is type I and contains a new edge and $\mathcal{V}(3, 4, 5)$ is type II and contains a Voronoi point that existed in V_1 and V_2.

III. Properties of the Order k Voronoi Diagrams

We now formalize our discussions given earlier and prove some properties of the order k Voronoi diagrams. Let us first introduce some notation.

The binary associative operation $\oplus$ on the set $I = \{1, 2, \cdots, n\}$ of indices is defined as follows. Let A and B be two subsets of I. $A \oplus B$ is defined as $(A - B) \cup (B - A)$, where "$-$" is the set subtraction. ($A \oplus B$ is commonly referred to as symmetric difference of two sets A and B.) For convenience we shall denote point q_i simply by i. Thus, the Voronoi polygon $\mathcal{V}(q_{i_1}, q_{i_2}, \cdots, q_{i_k})$ will be denoted by $\mathcal{V}(i_1, i_2, \cdots, i_k)$. Since the edges of the diagram are portions of bisectors of two points, we shall denote by $\overline{B}(i, j)$ the portion of bisector $B(i, j)$ that is an edge. The pair of indices (i, j) is said to be associated with the edge $\overline{B}(i, j)$. Let $\mathcal{V}_k(H, I)$ denote the polygon $\mathcal{V}_k(H)$ of the Voronoi diagram $V_k(I)$. In what follows, we shall use the set $I = \{1, 2, \cdots, n\}$ interchangeably with the set $S = \{q_1, q_2, \cdots, q_n\}$ of points.

Lemma 1: Given a set of points and a subset $H = \{i_1, i_2, \cdots, i_k\}$ of I, $k > 1$, if $H_s = H - \{j_s\}$, where $j_s \in H$, for $s = 1, 2, \cdots, m$, $m \leq k$, are the only subsets of H such that $\mathcal{V}_{k-1}(H_s, I)$ exist, then $\mathcal{V}_k(H, I)$ can be expressed as $\cup_{s=1}^{m}(\mathcal{V}_{k-1}(H_s, I) \cap \mathcal{V}_1(j_s, I - H_s))$.

Proof: Consider the order $k - 1$ diagram $V_{k-1}(I)$ and the intersection of $\mathcal{V}_k(H, i)$ and $V_{k-1}(I)$. Note that $\mathcal{V}_k(H, I) \cap \mathcal{V}_{k-1}(H', I) = \varnothing$ if $H' \neq H_s$, for some $s = 1, 2, \cdots, m$, for otherwise we can find a point q in the intersection such that the set of q's $(k - 1)$ nearest neighbors is not a proper subset of q's k-nearest neighbors which is impossible. Therefore, we have

$$\begin{aligned}\mathcal{V}_k(H, I) &= \mathcal{V}_k(H, I) \cap (\cup_{s=1}^{m}\mathcal{V}_{k-1}(H_s, I)) \\ &= \cup_{s=1}^{m}(\mathcal{V}_k(H, I) \cap \mathcal{V}_{k-1}(H_s, I)).\end{aligned}$$

Recall that $\mathcal{V}_k(H, I) = \cap h(H, I - H)$. We have

$$\begin{aligned}\mathcal{V}_k(H, I) \cap \mathcal{V}_{k-1}(H_s, I) &= (\cap h(H, I - H)) \cap (\cap h(H_s, I - H_s)) \\ &= (\cap h(j_s, I - H) \cap (\cap h(H_s, I - H))) \cap (\cap h(H_s, I - H_s)) \\ &= (\cap h(j_s, I - H)) \cap (\cap h(H_s, I - H) \cap (\cap h(H_s, I - H_s))) \\ &= \mathcal{V}_1(j_s, I - H_s) \cap (\cap h(H_s, I - H) \cap (\cap h(H_s, I - H_s))).\end{aligned}$$

Since $\cap h(H_s, I - H_s) = \cap h(H_s, I - H) \cap (\cap h(H_s, j_s)) \subseteq \cap h(H_s, I - H)$, $\cap h(H_s, I - H) \cap (\cap h(H_s, I - H_s)) = \cap h(H_s, I - H_s) = \mathcal{V}_{k-1}(H_s, I)$. Therefore, $\mathcal{V}_k(H, I) = \cup_{s=1}^{m}(\mathcal{V}_{k-1}(H_s, I) \cap \mathcal{V}_1(j_s, I - H_s))$. □

Lemma 2: Given an order k Voronoi diagram $V_k(I)$ if q is a point of an edge $\overline{B}(i, j)$ of $V_k(I)$, then the kth nearest neighbor of q is either i or j.[2]

Proof: It is obvious when $k = 1$. As an induction step consider any polygon $\mathcal{V}_{k-1}(H, I)$ and the diagram $V_1(I - H)$, i.e., the nearest neighbor diagram with the set H of points removed. Let q be a point in the intersection $\mathcal{V}_{k-1}(H, I) \cap V_1(I - H)$ such that q lies on an edge $\overline{B}(i, j)$ of $V_1(I - H)$. That is, $q \in \mathcal{V}_1(i, I - H)$ and $q \in \mathcal{V}_1(j, I - H)$. By Lemma 1, $q \in \mathcal{V}_k(H \cup \{i\}, I)$ and $q \in \mathcal{V}_k(H \cup \{j\}, I)$, which implies that q is on $\overline{B}(i, j)$ of $V_k(I)$. Since $q \in \mathcal{V}_{k-1}(H, I)$, H is the set of q's (k $-$ 1) nearest neighbors. Furthermore, q's nearest neighbor in the set $I - H$ of points is either i or j, it follows that q's kth nearest neighbor is either i or j. □

[2] In fact, both i and j are q's kth nearest neighbor. But either i or j is acceptable as q's kth nearest neighbor in our discussion.

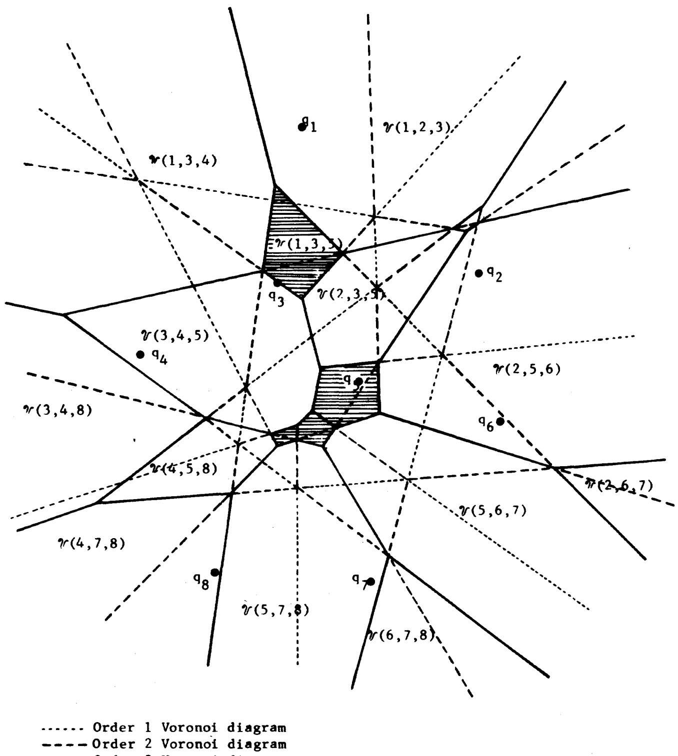

Fig. 6. The order 3, 2, and 1 Voronoi diagrams superimposed.

Lemma 3: Given an order k Voronoi diagram $V_k(I)$, q is a point of an edge $\overline{B}(I, j)$ of $V_k(I)$ if and only if the circle centered at q with radius $d(q, i) = d(q, j)$ contains $k - 1$ points in its interior.

Proof: If q is a point of $\overline{B}(i, j)$ of $V_k(I)$, then the claim is true by Lemma 2. To prove the converse consider the neighborhood $N(q, \delta)$ of q, i.e., a circle centered at q with radius δ. Since $q \in B(i, j)$ and there exists a circle K centered at q with radius $d(q, i) = d(q, j)$ which contains $k - 1$ points, by continuity of the distance metric we can find in $N(q, \delta)$ a point q' and a circle K' centered at q' with radius $d(q', i) = d(q', j)$ such that K' also contains the same $k - 1$ points. Let H denote the set of $k - 1$ points contained in K (and in K'). Since $q' \in B(i, j)$ is any point in $N(q, \delta)$, there exists a portion of $B(i, j)$ which lies in $N(q, \delta)$ such that for any point u on the portion of $B(i, j)$ the set of u's k-nearest neighbors is $H \cup \{i\}$ or $H \cup \{j\}$. This implies that the portion of $B(i, j)$ must be shared by $\mathcal{V}_k(H \cup \{i\})$ and $\mathcal{V}_k(H \cup \{j\})$ in $V_k(I)$. Thus, q is a point of $\overline{B}(i, j)$ of $V_k(I)$. □

Lemma 4: Given a set I of n points, the edges of the order $n-1$ Voronoi diagram $V_{n-1}(I)$ form a tree, i.e., for any two points on the edges there exists a unique path between them.

Proof: It follows from the fact that $V_{n-1}(I)$ is the farthest point Voronoi diagram and that all polygons of the farthest point Voronoi diagram are unbounded [11], [15], [21]. □

Lemma 5: The pair C of indices associated with an edge which borders two polygons $\mathcal{V}_k(A)$ and $\mathcal{V}_k(B)$ satisfies $C \oplus A \oplus B = \varnothing$. (This means that if one goes from one polygon to another, only the kth nearest neighbor is changed.)

Proof: Assume that $\overline{B}(i,j)$ borders $\mathcal{V}_k(A)$ and $\mathcal{V}_k(B)$. Consider a point q on $\overline{B}(i,j)$ and a neighborhood $N(q, \delta)$ of q. Let $q_1 \in \mathcal{V}_k(A) \cap N(q, \delta)$ and q_1 is not on $\overline{B}(i,j)$. Since q's kth nearest neighbor is either i or j, q_1's kth nearest neighbor must be i if δ is sufficiently small. [By the fact that $\lim_{\delta \to o} d(q_1, i) = d(q, i)$.] Therefore, A must be $A' \cup \{i\}$ such that A' is the set of q_1's $(k-1)$ nearest neighbors. Similarly, we can find a point $q_2 \in \mathcal{V}_k(B) \cap N(q, \delta)$ and q_2 is not on $\overline{B}(i,j)$ such that $B = B' \cup \{j\}$ and B' is the set of q_2's $(k-1)$ nearest neighbors. Since both q_1 and q_2 become q as δ approaches 0, we can conclude that q_1's and q_2's $(k-1)$ nearest neighbors must be identical, i.e., $A' = B'$. Therefore, $A \oplus B \oplus C = (A' \cup \{i\}) \oplus (A' \cup \{j\}) \oplus \{i, j\} = \varnothing$. □

With the above results we can formally classify the Voronoi points and edges of an order i Voronoi diagram as follows. Let $\overline{B}(a, b)$, $\overline{B}(b, c)$, and $\overline{B}(c, a)$ be edges of $V_i(S)$ incident upon a Voronoi point q such that $\overline{B}(a, b)$ borders $\mathcal{V}_i(H_1)$ and $\mathcal{V}_i(H_2)$, $\overline{B}(b, c)$ borders $\mathcal{V}_i(H_2)$ and $\mathcal{V}_i(H_3)$ and $\overline{B}(c, a)$ borders $\mathcal{V}_i(H_3)$ and $\mathcal{V}_i(H_1)$. q is said to be an *old* Voronoi point if $H_1 = H' \cup \{b, c\}$, $H_2 = H' \cup \{c, a\}$ and $H_3 = H' \cup \{a, b\}$, where $|H'| = i-2$ and H' does not contain a, b, or c; otherwise ($H_1 = H'' \cup \{a\}$, $H_2 = H'' \cup \{b\}$, and $H_3 = H'' \cup \{c\}$, where $|H''| = i-1$ and H'' does not contain a, b or c) q is said to be a *new* Voronoi point. A Voronoi edge is a *new* edge if it connects two new Voronoi points and is an *old* edge otherwise. Thus, an old edge can connect two Voronoi points that are old or one is new and the other is old. We now give some properties of the order k Voronoi diagram in terms of *old* and *new* Voronoi points and edges.

Lemma 6: Let $\overline{B}(a, b)$, $\overline{B}(b, c)$, and $\overline{B}(c, a)$ be edges of $V_k(S)$ incident upon a Voronoi point q. Let K_q denote the circle centered at q and passing through points a, b, and c. (K_q is the circumcircle of the triangle $\Delta a, b, c$.) If q is an old Voronoi point of $V_k(S)$, then K_q contains $k-2$ points of S in its interior. Otherwise, K_q contains $k-1$ points of S in its interior. (Note that this is only true under the assumption that no more than three points are cocircular.)

Proof: Suppose that $\mathcal{V}_k(H_1)$, $\mathcal{V}_k(H_2)$, and $\mathcal{V}_k(H_3)$ are the polygons sharing point q and $\overline{B}(a, b)$ borders $\mathcal{V}_k(H_1)$ and $\mathcal{V}_k(H_2)$, $\overline{B}(b, c)$ borders $\mathcal{V}_k(H_2)$ and $\mathcal{V}_k(H_3)$ and $\overline{B}(c, a)$ borders $\mathcal{V}_k(H_3)$ and $\mathcal{V}_k(H_1)$. If q is old, then by Lemma 2, K_q must contain the set H' of $k-2$ points in its interior. If q is new, then K_q contains the set H'' of $k-1$ points in its interior. □

Corollary 1: All Voronoi points of $V_1(S)$ are new.

Proof: Immediate. □

Lemma 7: Let $\mathcal{V}_k(H)$ be a *bounded* Voronoi polygon with vertices $u_1, u_2, \cdots, u_i$ $i \geq 3$. If all the Voronoi points $u_1, u_2, \cdots, u_i$ are new then k must be 1. If $k > 1$, then at least 2 of these i Voronoi points must be old.

Proof: Suppose that $\mathcal{V}_k(H_1), \mathcal{V}_k(H_2), \cdots, \mathcal{V}_k(H_i)$ are the polygons adjacent to $\mathcal{V}_k(H)$ such that u_j is shared by $\mathcal{V}_k(H_j)$, $\mathcal{V}_k(H_{j+1})$, and $\mathcal{V}_k(H)$ for $j = 1, 2, \cdots, i-1$ and u_i is shared by $\mathcal{V}_k(H_i)$, $\mathcal{V}_k(H_1)$, and $\mathcal{V}_k(H)$. 1) Suppose that $u_1, u_2, \cdots, u_i$ are new and $k \neq 1$. Since H and H_j can be expressed as $H = H_0 \cup \{a\}$, $H_j = H_0 \cup \{a_j\}$, $j = 1, 2, \cdots, i$, $|H_0| = k-1$ and $\overline{B}(a, a_j)$, $j = 1, 2, \cdots, i$ are edges of $\mathcal{V}_k(H)$, point a must be in the interior of $\mathcal{V}_k(H)$. Consider a point q on an edge of $\mathcal{V}_k(H)$. Let q' be the intersection of the line passing through q and a and the boundary of $\mathcal{V}_k(H)$. Let x be any point in H_0. Since point a, by Lemma 2, is the kth nearest neighbors of q and q', $d(q, x) < d(q, a)$ and $d(q', x) < d(q', a)$. Thus, $d(q, x) + d(q', x) < d(q, a) + d(q', a) = d(q, q')$, contradicting the triangle inequality $d(q, x) + d(q', x) \geq d(q, q')$. 2) Suppose that $k > 1$ and only one, say u_i, is an old Voronoi point. Since $u_1, u_2, \cdots, u_{i-1}$ are new, we have $H = H_0 \cup \{a\}$, $H_j = H_0 \cup \{a_j\}$, $j = 1, 2, \cdots, i$. But this implies that u_i, shared by H, H_1, and H_i must be new, a contradiction. □

This lemma implies that no new Voronoi points of an order $k > 1$ Voronoi diagram ever form a Voronoi polygon. By similar arguments, we have the following.

Corollary 2: No bounded order k Voronoi polygons have vertices that are all old.

Lemma 8: Given a Voronoi diagram $V_i(S)$, the new Voronoi points of $V_i(S)$ become old points of $V_{i+1}(S)$ and every polygon of $V_{i+1}(S)$ contains either a new edge of $V_i(S)$ (a type I region) or old points and old edges of $V_i(S)$ (a type II region) in its interior.

Proof: Consider two adjacent polygons $\mathcal{V}_i(H_1)$ and $\mathcal{V}_i(H_2)$ sharing an edge $\overline{B}(x, y)$ where $x \in H_1$ and $y \in H_2$. By Lemma 5, $H_1 \oplus H_2 = \{x, y\}$. The ith nearest neighbor of any point q on $\overline{B}(x, y)$ is either x or y by Lemma 2. Since the set of $(i+1)$ nearest neighbors of q on $\overline{B}(x, y)$ is $H_1 \cup \{y\} = H_2 \cup \{x\} = H_1 \cup H_2$, q must be in $\mathcal{V}_{i+1}(H_1 \cup \{y\})$. That is, each edge of $V_i(S)$ is contained in some polygon of $V_{i+1}(S)$.

Now suppose that q is a Voronoi point of $V_i(S)$ and the edges incident upon q are $\overline{B}(a, b)$, $\overline{B}(b, c)$, and $\overline{B}(c, a)$, respectively. Assume that the three polygons sharing q are $\mathcal{V}_i(H_1)$, $\mathcal{V}_i(H_2)$, and $\mathcal{V}_i(H_3)$ such that $\overline{B}(a, b)$ is shared by $\mathcal{V}_i(H_1)$ and $\mathcal{V}_i(H_2)$, $\overline{B}(b, c)$ shared by $\mathcal{V}_i(H_2)$ and $\mathcal{V}_i(H_3)$, and $\overline{B}(c, a)$ shared by $\mathcal{V}_i(H_3)$ and $\mathcal{V}_i(H_1)$. If q is old, then $\{a, b\} \subseteq H_3$, $\{b, c\} \subseteq H_1$ and $\{c, a\} \subseteq H_2$. By previous arguments, the edge $\overline{B}(b, c)$ is contained in $\mathcal{V}_{i+1}(H_3 \cup \{c\}) = \mathcal{V}_i{+}1(H_2 \cup \{b\})$, $\overline{B}(c, a)$ is contained in $\mathcal{V}_{i+1}(H_3 \cup \{c\}) = \mathcal{V}_{i+1}(H_1 \cup \{a\})$, and $\overline{B}(a, b)$ is contained in $\mathcal{V}_{i+1}(H_1 \cup \{a\}) = \mathcal{V}_{i+1}(H_2 \cup \{b\})$. Since $H_1 \cup \{a\} = H_2 \cup \{b\} = H_3 \cup \{c\}$, these three edges including point q belong to $\mathcal{V}_{i+1}(H_1 \cup \{a\})$ of $V_{i+1}(S)$. If q is new, then $a \in H_1$, $b \in H_2$, and $c \in H_3$. Now since $\overline{B}(a, b)$, $\overline{B}(b, c)$, and $\overline{B}(c, a)$ are contained, respectively, in $\mathcal{V}_{i+1}(H_1 \cup \{b\})$, $\mathcal{V}_{i+1}(H_2 \cup \{c\})$, and $\mathcal{V}_{i+1}(H_3 \cup \{a\})$, the point q will be shared by these three order $(i+1)$ polygons and become an old point of $V_{i+1}(S)$. Thus, if both Voronoi points q_1 and q_2 of an edge $\overline{B}(x, y)$ are new, i.e., $\overline{B}(x, y)$ is a new edge of $V_i(S)$, then $\overline{B}(x, y)$ will be contained in some polygon $\mathcal{V}_{i+1}(H)$ so that q_1 and q_2 are two *old* Voronoi points. This completes the proof. □

Lemma 9: Let $\mathcal{V}_i(H)$ be a Voronoi polygon of $V_i(S)$ and let $\mathcal{V}_1(j_1), \mathcal{V}_1(j_2), \cdots, \mathcal{V}_1(j_l)$ be the Voronoi polygons of $V_1(S - H)$ whose intersections with $\mathcal{V}_i(H)$ are nonempty. Then the Voronoi points of $V_1(S - H)$ that are in $\mathcal{V}_i(H)$ are new Voronoi points of $V_{i+1}(S)$ and the Voronoi edges of $V_1(S - H)$ that are in $\mathcal{V}_i(H)$ are Voronoi edges of $V_{i+1}(S)$. Furthermore, $\mathcal{V}_1(j_t) - \mathcal{V}_i(H) \neq \emptyset, t = 1, 2, \cdots, l$, i.e., no polygons of $V_1(S - H)$ lie entirely in $\mathcal{V}_i(H)$.

Proof: By Lemma 1, we have $\mathcal{V}_1(j_t) \cap \mathcal{V}_i(H) \subseteq \mathcal{V}_{i+1}(H \cup \{j_t\})$ for $t = 1, 2, \cdots, l$. That is, for any point $q \in \mathcal{V}_1(j_t) \cap \mathcal{V}_i(H)$, the set of q's $(i + 1)$ nearest neighbors is $H \cup \{j_t\}$. Since each of the Voronoi points of $V_1(S - H)$ that are in $\mathcal{V}_i(H)$ is shared by three polygons of the form $\mathcal{V}_{i+1}(H \cup \{j_s\})$, $\mathcal{V}_{i+1}(H \cup \{j_t\})$, and $\mathcal{V}_{i+1}(H \cup \{j_u\})$, these Voronoi points, according to our definition, are new. Since for any point q on an edge $\overline{B}(j_s, j_t)$ of $V_1(S - H)$ that is in $\mathcal{V}_i(H)$, the set of q's $(i + 1)$ nearest neighbors is $H \cup \{j_s\}$ or $H \cup \{j_t\}$, by Lemma 3, the portion of $\overline{B}(j_s, j_t)$ that is in $\mathcal{V}_i(H)$ is an edge of $V_{i+1}(S)$. To show that no polygons of $V_1(S - H)$ lie completely in $\mathcal{V}_i(H)$, observe that the point $j_t \notin \mathcal{V}_i(H)$ and $j_t \in \mathcal{V}_1(j_t)$, for $t = 1, 2, \cdots, l$, for if $j_t \in \mathcal{V}_i(H)$, the set of j_t's i nearest neighbors must include j_t which is not in H, a contradiction. □

Lemmas 8 and 9 show that new Voronoi points of $V_i(S)$ become old Voronoi points of $V_{i+1}(S)$ and the Voronoi points of $V_1(S - H)$ that are in $\mathcal{V}_i(H)$ are new Voronoi points of $V_{i+1}(S)$ and the Voronoi edges of $V_1(S - H)$ that are in $\mathcal{V}_i(H)$ are the edges of $V_{i+1}(S)$, for all $H \subseteq S$ whose associated polygon exists. Fig. 7 illustrates how type I and type II regions are formed from $V_i(S)$ to $V_{i+1}(S)$. We shall show below that the Voronoi points and edges of $V_{i+1}(S)$ are exactly those derived from $V_i(S)$ and $V_1(S - H)$ for all $H \subseteq S$ such that $\mathcal{V}_i(H) \neq \emptyset$.

Let $\mathcal{V}_{i+1}(H, S)$ denote any polygon of $V_{i+1}(S)$. Consider the Voronoi diagram $V_i(H)$, which partitions the plane into unbounded regions, each being the locus of points whose farthest neighbors in H are some extreme point of H. The edges of $V_i(H)$ will therefore partition $\mathcal{V}_{i+1}(H, S)$ into, say l, subregions, $R_1, R_2, \cdots, R_l$, each being the locus of points whose $(i + 1)$st nearest neighbors are some extreme point of H. The following claims are made: 1) the intersection of the edges of $\mathcal{V}_{i+1}(H, S)$ and the edges of $V_i(H)$ must occur at the vertices of $\mathcal{V}_{i+1}(H, S)$, which are *old* Voronoi points of $V_{i+1}(S)$; 2) the vertices of $\mathcal{V}_{i+1}(H, S)$ that are in some polygon $\mathcal{V}_i(H_t, H)$, $t \in \{1, 2, \cdots, l\}$, are *new* Voronoi points of $V_{i+1}(S)$; and 3) the intersection $\mathcal{V}_i(H_t, H) \cap \mathcal{V}_{i+1}(H, S)$ are exactly the intersection $\mathcal{V}_i(H_t, S) \cap \mathcal{V}_1(j_t, S - H_t)$, for $t = 1, 2, \cdots, l$, where $H_t = H - \{j_t\}$.

To prove 1) assume that an edge $\overline{B}(j_t, j_u)$ of $V_i(H)$ intersects an edge $\overline{B}(x, y)$ of $\mathcal{V}_{i+1}(H, S)$, $x \in H, y \notin H$, at a point q. Obviously, $d(q, j_t) = d(q, j_u) = d(q, y)$ and $x \in \{j_t, j_u\}$. That is, q is the circumcenter of $\Delta\, j_t j_u y$ and hence must be a vertex of $\mathcal{V}_{i+1}(H, S)$ shared by edges $\overline{B}(x, y)$ and $\overline{B}(x', y)$, where $\{x, x'\} = \{j_t, j_u\}$. Since the third edge incident with the vertex q is of the form $\overline{B}(x, x')$ and the polygons $\mathcal{V}_{i+1}(H_1, S)$ and $\mathcal{V}_{i+1}(H_2, S)$ that share $\overline{B}(x, y)$ and $\overline{B}(x', y)$ with $\mathcal{V}_{i+1}(H, S)$, respectively, are such that $H_1 \oplus H_2 = \{x, x'\} =$

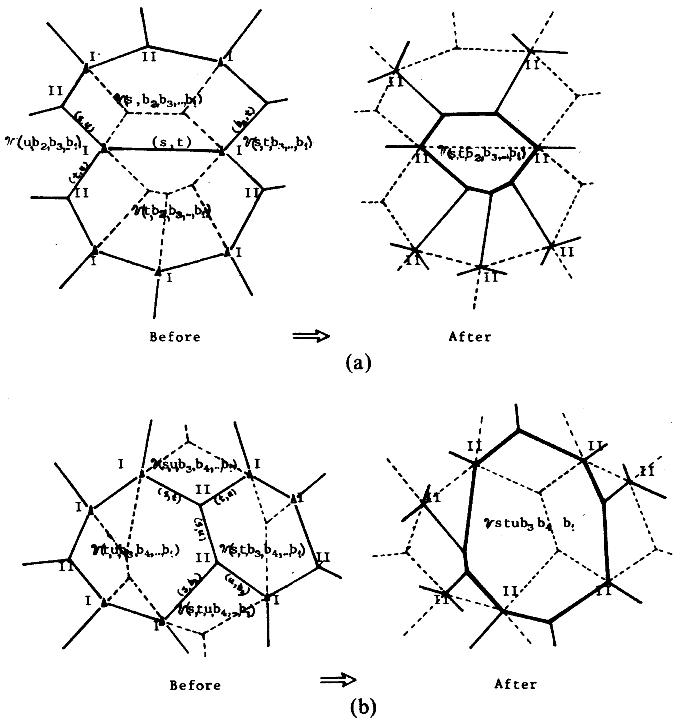

Fig. 7. Formation of type I and type II regions before and after the transformation from $V_i(S)$ to $V_{i+1}(S)$. (a) Formation of type I region. (b) Formation of type II region.

$\{j_t, j_u\} \subseteq H$, the point q is an old Voronoi point of $V_{i+1}(S)$. To prove 2) and 3), let $R_t = \mathcal{V}_i(H_t, H) \cap \mathcal{V}_{i+1}(H, S)$, $t = 1, 2, \cdots, l$. For any point $q \in R_t$ the set of q's i nearest neighbors is H_t and the set of q's $(i + 1)$st nearest neighbor is $\{j_t\}$. This implies that $R_t \subseteq \mathcal{V}_i(H_t, S)$ and $R_t \subseteq \mathcal{V}_1(j_t, S - H_t)$. Therefore, $R_t \subseteq \mathcal{V}_i(H_t, S) \cap \mathcal{V}_1(j_t, S - H_t)$. But from Lemma 1, $\mathcal{V}_i(H_t, S) \cap \mathcal{V}_1(j_t, S - H_t) \subseteq \mathcal{V}_{i+1}(H, S)$ and since $\mathcal{V}_i(H_t, S) \subseteq \mathcal{V}_i(H_t, H)$, we have $\mathcal{V}_i(H_t, S) \cap \mathcal{V}_1(j_t, S - H_t) \subseteq \mathcal{V}_{i+1}(H, S) \cap \mathcal{V}_i(H_t, H) = R_t$. Thus, $\mathcal{V}_i(H_t, H) \cap \mathcal{V}_{i+1}(H, S) = \mathcal{V}_i(H_t, S) \cap \mathcal{V}_1(j_t, S - H_t)$. The vertices of $\mathcal{V}_{i+1}(H, S)$ that are in $\mathcal{V}_i(H_t, H)$ must be shared by edges of the form $\overline{B}(j_t, y)$ and $\overline{B}(j_t, y')$ where $y, y' \in S - H$, since for any point on the edges its $(i + 1)$st nearest neighbor is j_t. Since the third edge incident with any such vertex q is of the form $B(y, y')$ and the polygons $\mathcal{V}_{i+1}(H_1, S)$ and $\mathcal{V}_{i+1}(H_2, S)$ that share $\overline{B}(j_t, y)$ and $\overline{B}(j_t, y')$ with $\mathcal{V}_{i+1}(H, S)$, respectively, are such that $H_1 \oplus H_2 = \{y, y'\} \not\subseteq H$, it follows that q is a new Voronoi point of $V_{i+1}(S)$. In fact, those edges of $\mathcal{V}_{i+1}(H, S)$ that are in $\mathcal{V}_i(H_t, H)$ are edges of $\mathcal{V}_1(j_t, S - H_t)$ and hence are edges of $V_1(S - H)$; those edges of $\mathcal{V}_i(H_t, H)$ that are in $\mathcal{V}_{i+1}(H, S)$ are edges of $\mathcal{V}_i(H, S)$.

Thus we have the following theorem.

Theorem 1: The Voronoi diagram $V_{i+1}(S)$ can be obtained from $V_i(S)$ and $V_1(S - H)$ by taking intersections of $\mathcal{V}_i(H)$ and $V_1(S - H)$ for all $H \subseteq S$ such that $\mathcal{V}_i(H)$ is nonempty.

Lemma 10: If $\mathcal{V}_i(H)$ contains m old Voronoi points of $V_{i-1}(S)$ in its interior then it also contains $2m + 1$ old edges of $V_{i-1}(S)$.

Proof: Consider the intersection of $\mathcal{V}_i(H)$ and $V_{i-1}(S)$. By Lemma 1, $\mathcal{V}_i(H, S) = \cup_{s=1}^{t}(\mathcal{V}_{i-1}(H_s, S) \cap \mathcal{V}_1(j_s, S - H_s))$, where $H_s = H - \{j_s\}$ and $j_s \in H$ for $s = 1, 2, \cdots, t$. This means that $\mathcal{V}_i(H, S)$ is the union of t regions in which each region is the intersection of the locus of points whose $(i-1)$ nearest neighbors are the points in H_s and the locus of points whose ith nearest neighbor is j_s. In other words, each region is the locus of points whose i-nearest neighbors are in the set $H_s \cup \{j_s\} = H$ and ith nearest neighbor is j_s. Consider now the diagram $V_{i-1}(H)$. Since $V_{i-1}(H)$ partitions the polygon $\mathcal{V}_i(H, S)$ into exactly the regions each of which R_{j_s} is the locus of points whose i-nearest neighbors are the points in H and the ith nearest neighbor is j_s, each region R_{j_s} is identical to $\mathcal{V}_{i-1}(H_s, S) \cap \mathcal{V}_1(j_s, S - H_s)$. By Lemma 4 the edges that partition $\mathcal{V}_i(H, S)$ form a tree. Therefore, if there are m points then there are $2m + 1$ edges. □

Lemma 11: Let N_k, E_k, I_k, and $\mathcal{S}_k$ denote, respectively, the numbers of polygons, edges, Voronoi points, and unbounded regions of an order k Voronoi diagram $V_k(S)$. The following relations hold:

$$E_k = 3(N_k - 1) - \mathcal{S}_k \text{ and } I_k = 2(N_k - 1) - \mathcal{S}_k.$$

Proof: This becomes straightforward if one considers the dual graph $D_k(S)$ in which each node of $D_k(S)$ corresponds to a polygon of $V_k(S)$, and two nodes of $D_k(S)$ are connected by an edge if the two corresponding polygons share an edge. Note that the number of edges, faces (triangles due to the assumption that each Voronoi point has degree 3), and nodes which bound the outer infinite face of $D_k(S)$ are equal, respectively, to the numbers of Voronoi edges, Voronoi points, and unbounded regions. □

Theorem 2: Given a set S of N points, the total number of regions N_k in $V_k(S)$ is $N_k = (2k-1)N - (k^2 - 1) - \sum_{i=1}^{k} \mathcal{S}_{i-1}$, for $N \geq k \geq 1$, where $\mathcal{S}_i$ is the number of unbounded regions in $V_i(S)$ and $\mathcal{S}_0$ is defined to be 0.

Proof: Let E_i and I_i denote, respectively, the number of edges and Voronoi points in $V_i(S)$. Let I'_{i+1} denote the number of new points in V_{i+1}, i.e., the number of points that do not exist in $V_i(S)$. I'_i is equal to I_1. Then from the above arguments we have

$$I_{i+1} = I'_{i+1} + I'_i \tag{1}$$

since the number of Voronoi points in $V_{i+1}(S)$ is the sum of the number of new points in $V_{i+1}(S)$ and the number of old points in $V_{i+1}(S)$ which were new in $V_i(S)$. Rearranging (1), we have

$$I'_{i+1} = I_{i+1} - I'_i. \tag{2}$$

Now let E'_{i+1} and E''_{i+1} be the number of new edges connecting new Voronoi points and the number of edges incident upon old Voronoi points in $V_{i+1}(S)$, respectively. We have $E'_1 = E_1$ and E_1 and

$$E_{i+1} = E'_{i+1} + E''_{i+1}.$$

Since each new edge corresponds to a type I region, the total number N'_{i+2} of type I regions in $V_{i+2}(S)$ is E'_{i+1}. Each type II region in $V_{i+2}(S)$ contains old Voronoi points in $V_{i+1}(S)$. Suppose that the total number of type II regions in $V_{i+2}(S)$ is N''_{i+2}, and the jth type II region contains m_j old Voronoi points. We have $\sum_{j=1}^{N''_{i+2}} m_j = I'_i$. Also, the number of edges e_j incident on these m_j points is, by Lemma 10, $e_j = 2m_j + 1$ and $\sum_{j=1}^{N''_{i+2}} e_j = E''_{i+1}$. Therefore, we have

$$E''_{i+1} = 2 \sum_{j=1}^{N''_{i+2}} m_j + N''_{i+2} = 2I'_i + N''_{i+2}$$

or

$$N''_{i+2} = E''_{i+1} - 2\,I'_i.$$

Thus

$$\begin{aligned} N_{i+2} &= N'_{i+2} + N''_{i+2} \\ &= E'_{i+1} + E''_{i+1} - 2\,I'_i \\ &= E_{i+1} - 2\,I'_i. \end{aligned} \tag{3}$$

By Lemma 11 and (2) and (3) we have the following recurrence relation for N_k:

$$N_1 = N, N_2 = E'_1 = 3(N - 1) - \mathcal{S}_1$$

$$N_{k+2} = 3(N_{k+1} - 1) - \mathcal{S}_{k+1} - 2\sum_{i=1}^{k} (-1)^{k-i}(2(N_i - 1) - \mathcal{S}_i).$$

By induction on k we can prove that

$$N_k = (2k-1)N - (k^2 - 1)N - \sum_{i=1}^{k} \mathcal{S}_{i-1}. \tag{4}$$ □

Corollary 3: The total number of regions in $V_k(S)$ is $0(k(N - k))$.

Proof: It follows directly from (4). □

Lemma 12: The number of new Voronoi points in $V_k(S)$ is

$$I'_k = 2k(N-1) - k(k-1) - \sum_{i=1}^{k} \mathcal{S}_i.$$

Proof: By Lemma 11 and (2) and (4) the lemma can be verified easily.

Next we shall also derive two interesting results which have been obtained by Shamos and Hoey [22].

Lemma 13: The total number of bounded and unbounded regions in the Voronoi diagram of all orders is $\sum_{k=1}^{N-1} B_k = \binom{N-1}{3}$ and $\sum_{k=1}^{N-1} \mathcal{S}_k = N(N-1)$, respectively.

Proof: Since $N_N = 1$, i.e., the number of regions in $V_N(S)$ is 1, which is the plane itself, by Theorem 2 we have $\sum_{k=1}^{N-1} \mathcal{S}_k = N(N-1)$. By Lemma 11, $I_k = 2(N_k - 1) - \mathcal{S}_k$ and the fact that $\sum_{k=1}^{N-1} I_k = 2\binom{N}{3}$, for each Voronoi point which is the circumcenter of some 3 points of S appears twice in the Voronoi diagram of all orders, we have

$$\begin{aligned} N - \sum_{k=1}^{1} B_k &= \sum_{k=1}^{N-1} (N_k - \mathcal{S}_k) = \frac{1}{2}\sum_{k=1}^{N-1} (I_k - \mathcal{S}_k + 2) \\ &= \binom{N}{3} - \frac{1}{2}N(N-1) + (N-1) = \binom{N-1}{3}. \end{aligned}$$

Q.E.D.

IV. The Algorithm

In this section we shall give an algorithm to partition a single Voronoi polygon, say $\mathcal{V}(q_m)$, of the order 1 Voronoi diagram $V_1(S)$. Suppose that the polygons $\mathcal{V}(q_0), \cdots, \mathcal{V}(q_{n-1})$ are adjacent to polygon $\mathcal{V}(q_m)$. We want to partition $\mathcal{V}(q_m)$ into n subregions such that each subregion r_i is the locus of points closer to q_i than to any other points except q_m for $i = 0, 1, \cdots, n-1$ (cf. Theorem 1). As pointed out in Section II, to partition $\mathcal{V}(q_m)$ is essentially equivalent to artificially deleting q_m and forming the Voronoi diagram for the remaining points $q_0, q_1, \cdots, q_{n-1}$. The effect of deleting q_m is the elimination of the boundary of the polygon and the extension of the edges intersecting at the vertices of $\mathcal{V}(q_m)$ to the interior of $\mathcal{V}(q_m)$, thereby partitioning the polygon $\mathcal{V}(q_m)$. Let the vertices of $\mathcal{V}(q_m)$ be denoted as $I_o, I_1, \cdots, I_{n-1}$ and be kept as a linked list. Each edge (I_i, I_{i+1})[3] of the polygon is a portion of the bisector $B(q_i, q_m)$ and is represented by the index pair (i, m). By assumption each vertex I_i is an intersection of three edges represented by (i, m), $(i-1, m)$, and $(i, i-1)$. Note that the index pairs of the three edges satisfy the property $(a, b) \oplus (b, c) \oplus (c, a) = \emptyset$. Let us denote the edge which is incident with I_i and which is not on the boundary of the polygon $\mathcal{V}(q_m)$ by IND(I_i). If the two boundary edges which are incident with I_i are represented by (a, b) and (a, c), then IND $(I_i) = (a, b) \oplus (a, c) = (b, c)$. Fig. 8 shows a typical Voronoi polygon $\mathcal{V}(10)$ which is to be partitioned.

We shall tackle this problem by divide-and-conquer technique. We first obtain the Voronoi diagram for sets of 3 points $\{q_{n-1}, q_o, q_1\}$, $\{q_2, q_3, q_4\}$, etc., by extending the edges associated with {IND (I_0), IND (I_1)}, {IND (I_3), IND (I_4)}, etc., respectively, into the interior of $\mathcal{V}(q_m)$. By "merging" two adjacent Voronoi diagrams for sets of 3 points, we get Voronoi diagrams for sets of 6 points. Repeating this merge process $\lceil \log_2 n/3 \rceil$ times, we will obtain the Voronoi diagram for n points. The edges of the diagram which are interior to $\mathcal{V}(q_m)$ will partition $\mathcal{V}(q_m)$ into n subregions. Fig. 9 shows the merge process of two Voronoi diagrams for $\{q_9, q_o, q_1\}$ and $\{q_2, q_3, q_4\}$, and Fig. 10 shows the final merge process of two Voronoi diagrams for $\{q_9, q_o, q_1, \cdots, q_4\}$ and $\{q_5, q_6, \cdots, q_9\}$. In Fig. 10 the merge process starts with the extension of IND (I_5), i.e., the bisector (4, 5) and ends at the point E as shown.

The technique used to merge two Voronoi diagrams is discussed in detail in [13], [14], and [22]. Here we shall omit the description of the merge process and describe a method of identifying the set of Voronoi points on which the divide-and-conquer technique is to be applied.

Note that the result of Lemma 7 is crucial to the correctness of the merging algorithm since it ensures that the "merge lines" in the merging process always consist of no cycles.

Suppose that we have obtained an order i Voronoi diagram $V_i(S)$, $k > i \geq 1$. Recall that there are two types of Voronoi points, old points which existed in $V_{i-1}(S)$ and new points which are just created. Based on previous discussions, only the set of new Voronoi points are needed in order to construct the $V_{i+1}(S)$ diagram. Assume inductively that each Voronoi polygon $\mathcal{V}_i(H_j)$ is associated with i points, i.e., $\mathcal{V}_i(H_j)$ is the locus of points closer to these i points than to the remaining points of S and its vertices are represented as a double-linked circular list. Since some of the vertices of $\mathcal{V}_i(H_j)$ are old Voronoi points and some of them are new Voronoi points, only those new Voronoi points are further linked together using an additional pointer. Note that all the vertices in $V_1(S)$ are new Voronoi points. Now to partition $\mathcal{V}_i(H_j)$ we first obtain the set of new Voronoi points, each of which is associated with an edge represented by IND (I_j) where I_j is a new Voronoi point of $\mathcal{V}_i(H_j)$, and then apply the divide-and-conquer technique just described. After each Voronoi polygon is partitioned we need to chain together the vertices of each newly created Voronoi polygon, associate it with a new set of $i+1$ points, and

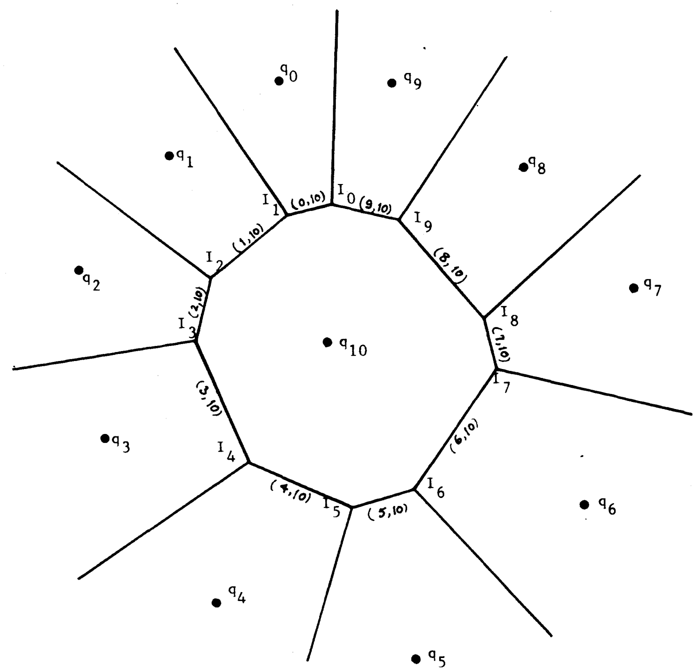

Fig. 8. The order 1 Voronoi diagram for a set of 11 points.

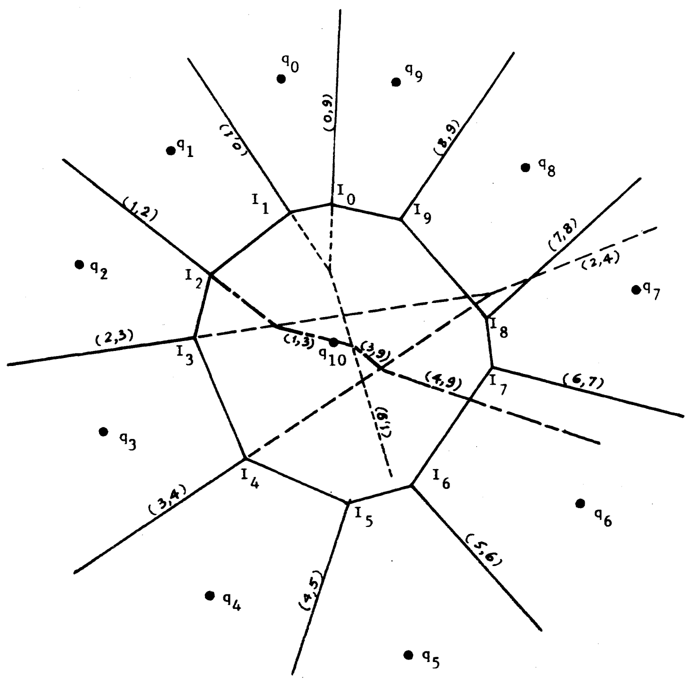

Fig. 9. Illustration of the merging of two Voronoi diagrams for a set of 3 points.

[3] Subscript additions and subtractions are taken modulo n.

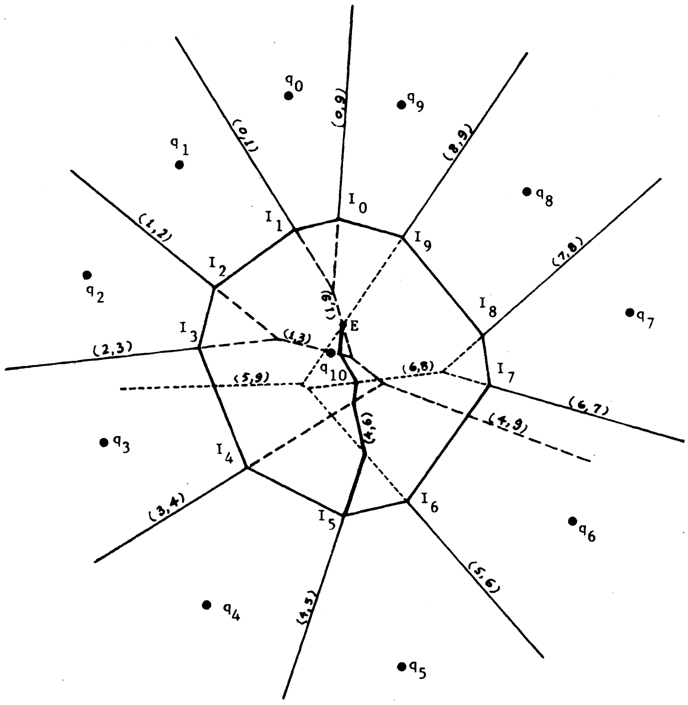

Fig. 10. Final merge process to form the Voronoi diagram for a set of 10 points.

at the same time when we form the vertex list of each polygon we link the new vertices of the polygon to form a list. In this manner we can obtain the order $i + 1$ diagram $V_{i+1}(S)$.

Now let us analyze the running time of the algorithm. Suppose that the polygon $\mathcal{V}_i(H)$ to be partitioned has s new Voronoi points $I_1, I_2, \cdots, I_s$. The process of partitioning $\mathcal{V}_i(H)$ is equivalent to finding the edges of an order 1 Voronoi diagram for the set of s points whose indices are the union of those that are associated with the edges IND $(I_i), i = 1, 2, \cdots, s$, and therefore takes $0(s \log s)$ time. Since there are N_i Voronoi polygons in $V_i(S)$ and $0(iN)$ new Voronoi points (Lemma 12), the total number of operations required is

$$\sum_{j=1}^{N_i} 0(s_j \log s_j) = 0\,(i\,N \log N).$$

The amount of storage is $0(N_{i+1}) = 0((i + 1)(N - (i + 1)))$. Since $k - 1$ iterations are required to obtain an order k Voronoi diagram, the entire work is $\Sigma_{i=1}^{k-1} 0(i\,N \log N) = 0(k^2 N \log N)$. The amount of storage required is $0(k \cdot N_k)$, which is $0(k^2(N - k))$, since there are $0(N_k)$ regions and each region is associated with K points.

V. Search Algorithms for k-Nearest Neighbors

After we have constructed the order k Voronoi diagram for N sample points, to find the k-nearest neighbors of a given test point X (or a set of test points) we simply perform a point-location operation to locate the region which contains the test point. There are a number of algorithms which have appeared in the literature. We shall summarize here the results in Table I without going into detailed description of the algorithms. The interested readers can consult the original articles given in the reference.

TABLE I

Scheme	Preprocessing Time	Storage	Search Time
1.Slab method [6,22]	$0(N_k^2)$	$0(N_k^2)$	$0(k + \log N_k)$
2.Lee-Preparata [16]	$0(N_k \log N_k)$	$0(kN_k)$	$0(k + (\log N_k)^2)$
3.Lipton-Tarjan* [17]	$0(N_k \log N_k)$	$0(k N_k)$	$0(k + \log N_k)$
4.Preparata [19]	$0(N_k \log N_k)$	$0(kN_k + N_k \log N_k)$	$0(k + \log N_k)$
5.Kirkpatrick [12]	$0(N_k \log N_k)$	$0(k N_k)$	$0(k + \log N_k)$

*According to Lipton and Tarjan, this method is far from being practical.

VI. Concluding Remarks

We have presented an $0(k^2 N \log N)$ algorithm to construct the order k Voronoi diagram for a set of N points in the Euclidean plane using $0(k^2(N - k))$ storage.

As noted in Section II, the final Voronoi diagram is useful only when k is fixed. In the case that k can vary over a certain range, we can either keep a copy of Voronoi diagram of order k for each possible value of k, or construct the Voronoi diagram whose order is the largest k; and among the k-nearest neighbors found we choose the k' nearest ones if we only are interested in k' nearest neighbors. By using the former simple-minded method, we need $\sum_k 0(k^2(N - k))$ storage, where summation is taken over k within the specified range, and $0(k_{\max}^2 N \log N)$ preprocessing time. By the latter approach, we can save storage but have to pay for the time spent in selecting k-nearest neighbors among $k_{\max}$ nearest ones. But this only takes $0(k_{\max})$ operations by using the linear selection algorithm of Blum *et al.* [3]. In addition, there is a minor point that is worth mentioning. Since finding the k-nearest neighbors among N points is equivalent to find $(N - k)$ farthest neighbors, we can restrict k to be upper bounded by $\left\lceil \frac{N}{2} \right\rceil$. The constructions of the farthest neighbor Voronoi diagrams and the generalized order K farthest neighbor diagrams are very similar to the constructions of their counterparts described in the paper. For construction of the farthest neighbor Voronoi diagrams, see [11], [15], and [21]. Furthermore, if k could be as large as $0(N)$, we do not gain much by elaborating on such an expensive restructuring on the set of points.

The order k Voronoi diagram can be useful if one is interested in finding for each of the N points its k-nearest neighbors. A brute force method would take $0(N^2)$ time. However, if we construct the order $(k + 1)$ Voronoi diagram $V_{k+1}(S)$ and then apply the point-location algorithms (methods 3–5) to locate these N data points in $V_{k+1}(S)$, we can solve the all-k-nearest neighbor problem in time $0(kN_k \log N_k)$, which is $0(k^2 N \log N)$. Thus, this technique is useful only when $k < 0((N/\log N)^{1/2})$. In addition, since each of those "new" Voronoi points in the order k Voronoi diagram corresponds to a circumcenter of some three points and the circumcircle thus determined contains exactly $k - 1$ data points in its interior, the problem of finding a smallest circle that contains $k - 1$ points in the interior can be solved in $0(k(N - k))$ time after the diagram is obtained.

We have made the assumption that no more three points are cocircular. In case there are at most $d(d \geqslant 3)$ points that are cocircular, some results that we have obtained must be modified. For example, the claim that the circle centered at a Voronoi point q of $V_k(S)$ and passing through three points contains either $k-1$ or $k-2$ points in its interior should be modified so that it may contain $k-1, k-2, \cdots$, or $k-(d-1)$ points in its interior. However, the central result (Theorem 1) that the Voronoi diagram $V_{i+1}(S)$ can be obtained from $V_i(S)$ and $V_1(S-H)$ for all $\mathcal{V}_i(H) \neq \emptyset$ and $H \subseteq S$, still holds, except that the notion of new and old Voronoi points must be changed. Let $\mathcal{V}_i(H)$ be a Voronoi polygon and $\overline{B}(x, y)$ and $\overline{B}(x', z)$ be two edges of $\mathcal{V}_i(H)$ that meet at a vertex q, where $x, x' \in H$. Then q is old (and will vanish in $V_{i+1}(S)$) if $y = z$ and q is new (and will remain new for $d-2$ iterations in $V_{i+1}(S), \cdots, V_{i+d-2}(S)$, where d is the number of points that are on the circle centered at q) if $y \neq z$. Note that this classification becomes what we had earlier when $d = 3$. In case q is new, a portion of the bisector $B(y, z)$ will belong to $V_{i+1}(S)$ (as opposed to IND (q) when $d = 3$).

The construction of the Voronoi diagram of order k presented here is based on an iterative approach, i.e., transforming an order j Voronoi diagram to order $j+1$ diagram with one iteration. Does there exist a better way for constructing an order k diagram? As one may have noticed, in the course of building up an order $j+1$ diagram from an order j diagram, many Voronoi points are created and then destroyed in the following iteration. Therefore, it is reasonable to ask whether there is any short cut that leads to the order k diagram without going through every stage

One other possible approach is to apply the divide-and-conquer technique directly to the set of points to get the order k Voronoi diagram in a way similar to the one used in constructing the nearest neighbor Voronoi diagram. The difference between these two is that in the latter case there is only one polygonal line, but in the former case there will be a set of polygonal lines. The problem thus resides in how to obtain such a set of polygonal lines.

Finally, the generalization of the idea of Voronoi diagram in L_2-metric to L_p-metric for $1 \leq p \leq \infty$, is straightforward. The iterative algorithm presented is applicable for constructing the order k Voronoi diagram in L_p-metric. Interested readers are referred to [14].

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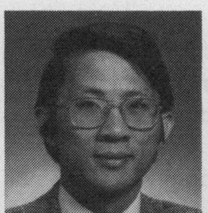

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